International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

Nonlinear consensus via continuous, sampled, and aperiodic updates

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Version of record first published: 14 Jan 2013.

International Journal of Control
2013, 1–12, iFirst

Nonlinear consensus via continuous, sampled, and aperiodic updates

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(*Received 2 April 2012; final version received 6 November 2012*)

We consider a first-order multi-agent system with nonlinear control protocols performing consensus. We first address the convergence properties of the continuous system. Then periodically sampled control inputs are treated, where we derive explicit upper bounds on the sampling interval to preserve global stability. Moreover, we design an aperiodic and event-triggered updating law to reduce the control efforts even further while ensuring the closed-loop stability and providing a strictly positive lower bound on the inter-execution time. Finally, the robustness of this approach with respect to additive disturbances is examined by applying $\mathcal{L}_2$ gain analysis.

Keywords: multi-agent consensus; nonlinear system; sampled control; event-triggered control; $\mathcal{L}_2$ stability

1. Introduction

Consensus is one of the most studied applications in the multi-agent control area, in which a collection of agents aim at agreeing upon certain quantities of interest. The primary goal is to design a distributed control law for each agent such that consensus is achieved at steady state. The well-known linear consensus protocol using convex combination of relative states between neighbouring agents as control parameters has been evaluated under static, switching communication topologies and under communication delays in Olfati-Saber and Murray (2004), over limited and unreliable communication in Ren and Beard (2005). Synchronisation in networks of identical linear systems instead of single integrator agents, is considered in Scardovia and Sepulchre (2009).

Recent literature extensively addresses the consensus problem with additional design constraints like connectivity preservation in Olfati-Saber (2006), finite time convergence in Cortés (2006), Ren, Cao, and Meng (2010), and quantised information in Ceragioli, De Persis, and Frasca (2010), Guo and Dimarogonas (2011). Nonlinear consensus protocols arise naturally in most of the above applications. In Ajorlou, Momeni, and Aghdam (2011) and Chen, Liua, and Lua (2009), the dynamical agents performing consensus are inherently described by nonlinear dynamics. Nonlinearities of the control protocol increase the difficulties of stability analysis due to the fact that it is not possible to write the closed-loop system as $\dot{x} = -Lx$ and further utilise the spectrum properties of the Laplacian matrix L as in Olfati-Saber and Murray (2004). While most of the aforementioned papers tackle the consensus problem under nonlinear protocols, we explore here not only the stability of the continuous model but also the discrete-time counterparts due to sampled or event-triggered control updates.

On the other hand, since continuous communication is an ideal assumption, it is more realistic to assume that the participants can only interact at discrete time instants. One choice is to use periodic and synchronous clock for all agents so that they communicate and update the control law synchronously at the same sampling instants, as analysed in Xie, Liu, Wang, and Jia (2009). It is mentioned that the value of this sampling period greatly affects the stability and performance of the resulting discrete-time system. In case of large networks, an increasing number of agents lead to a demand for reduced computation and limited communication bandwidth per agent. In that respect, an event-triggered approach seems more favourable for multi-agent systems. The application of event-triggered strategy to multi-agent consensus problem can be found in De Persis, Sailer, and Wirth (2011), Dimarogonas and Johansson (2009), Dimarogonas, Frazzoli, and Johansson (2012), Seyboth, Dimarogonas, and Johansson (2011) and Wang and Lemmon (2011). Both stochastic event-triggered strategies in Rabi, Johansson, and Johansson (2008) and deterministic event-triggered strategies in Wang and Lemmon (2011) have been considered. Moreover, it is important to guarantee that all agents had a strictly nonnegative inter-event time in order to

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ISSN 0020–7179 print/ISSN 1366–5820 online

http://dx.doi.org/10.1080/00207179.2012.747735
http://www.tandfonline.com

avoid infinitely often triggering and also the robustness of the proposed control law with respect to additive noise in the system model as discussed in Dimarogonas (2011).

The main contribution of this work can be summarised as follows: we take into account the first-order multi-agent system with a generic class of nonlinear control protocols. We first examine the convergence of the continuous system and then extend the results to the same system with periodically sampled inputs. Explicit upper bounds on sampling intervals are derived to ensure global convergence. Moreover, we design a decentralised event-triggered law that substantially reduces both the control efforts and the communication efforts. At the same time, the case of infinitely frequent triggering is excluded by showing that the inter-event interval is strictly larger than a positive lower bound. Moreover the robustness of this method with respect to additive disturbances is examined by applying $\mathcal{L}_2$ gain analysis in the last part.

Compared with the event-triggered strategies proposed in Dimarogonas et al. (2012) and Dimarogonas (2011), we provide an approach that allows for both event-triggered communication and event-triggered control updates. Each agent broadcasts its state information to its neighbours only at specific event-based instants and the control law is updated, whenever the agent receives new measurements. In addition, the designed triggering condition is piecewise constant, which means that continuous monitoring of neighbours' states is avoided. Similar arguments can be found in Seyboth et al. (2011), where event-based broadcasting is discussed, but this article treats a more general problem by taking into account nonlinear consensus protocols, under not only event-triggered control updates, but also continuous and periodically sampled control updates. Furthermore, the event-triggered updating law proposed in this article is purely decentralised in the sense that no global knowledge is needed like the overall communication structure required in Seyboth et al. (2011).

The rest of this article is organised as follows: Section 2 presents some necessary backgrounds from algebraic graph theory and introduces the system dynamics we consider. In Section 3, we investigate the stability of the continuous system and derive the convergence rate under static and switching communication topologies. Section 4 is devoted to the same multi-agent system but with periodically sampled control inputs. The decentralised and event-triggered control strategy is discussed in Section 5, followed by the $\mathcal{L}_2$ gain robustness analysis for the case of additive noise in each agent's dynamics in Section 6. Some numerical simulations are given in Section 7 while the last section concludes the results and indicates further research directions.

2. Problem statement

2.1 *Graph theory and consensus preliminaries*

First, we present some definitions and notations from algebraic graph theory in Godsil and Royle (2001). For an undirected graph $G=(\mathcal{V}, E)$ with N vertices, denote by $\mathcal{V}=1,\ldots,N$ the set of *vertices* and by $E=\{(i,j)\in\mathcal{V}\times\mathcal{V}|i\in\mathcal{N}_j\}$ the set of *edges*, where $\mathcal{N}_j$ denotes agent j's *communication set* that includes the agents with which it can communicate. Each agent only has access to the state of agents that belong to its communication set. In the sequel, G is assumed to be *undirected*, namely $i\in\mathcal{N}_j\Leftrightarrow j\in\mathcal{N}_i$, $\forall(i,j)\in E$. When the communication topology is static, G is time-invariant and the sets $\mathcal{N}_i$ are static. Otherwise $G=G(t)$ if the neighbouring sets $\mathcal{N}_i$ change over time.

In this article, we mainly consider weighted and undirected graphs, for which the *adjacency matrix* $A=A(G)=\{a_{ij}\}$ is the $N\times N$ matrix given by $a_{ij}>0$, if $(i,j)\in E$, and $a_{ij}=0$, otherwise. Agents i, j are called *adjacent* if $(i,j)\in E$. A *path* of length r from i to j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices, then G is called *connected*. A connected graph is called a *tree* if it contains no cycles. Let Δ be the $N\times N$ diagonal matrix of d_i's, where the *degree* d_i of each vertex i is given by $d_i=\sum_{j=1}^{N}a_{ij}$. The *Laplacian* matrix of G is the symmetric positive semidefinite matrix given by $L=\Delta-A$. For a connected graph, L has non-negative eigenvalues, Godsil and Royle (2001) and a single zero eigenvalue with the corresponding eigenvector $\mathbf{1}$, where $\mathbf{1}=(1,\ldots,1)^T$. Denote by $\lambda_k(M)$ the kth eigenvalue of matrix M in ascending order, then $\lambda_1(L)=0$ and $\lambda_2(L)>0$. Note here, we do not restrict a_{ij} to be uniformly one or equal, which means the edges can have different weights.

2.2 *System model*

Consider N single-integrator agents:

$$\dot{x}_i=u_i,\quad i\in\mathcal{V},\tag{1}$$

where $x_i\in\mathbb{R}$ denotes the state and $u_i\in\mathbb{R}$ the control input of agent i. The control objective is to construct distributed feedback controllers such that all agents converge to an agreement state. In this article, we treat only the system behaviour in the x-coordinates but the analysis that follows applies directly in higher dimensions. Denote by $\mathbf{x}=[x_1,\ldots,x_N]^T$ the stack vector for the agents' coordinates in x-coordinates.

We consider the following nonlinear consensus protocol for system (1):

$$u_i = -\sum_{j=1}^{N} a_{ij} f(x_i - x_j), \quad i \in \mathcal{V}, \tag{2}$$

where it is assumed that $a_{ij} = a_{ji} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise. It means that we only take into account undirected topologies with symmetric weights in this article. The scalar function $y = f(x)$: $\mathbb{R} \to \mathbb{R}$ and $f \in \mathbb{C}^1$, has the following properties:

- $f(x) = 0$ if and only if $x = 0$
- $f(x) = -f(-x)$, $\forall x \in \mathbb{R}$
- $\gamma_1\, x \le f(x) \le \gamma_2\, x$, where $\gamma_2 > \gamma_1 > 0$, $\forall x \in \mathbb{R}^+$.

According to the definitions in Chapter 6, Khalil (2002), the above scalar function $y = f(x)$ is *input-feedforward passive* and *output strictly passive*. Figure 1 illustrates an example of function $f(x)$ and also the sector $[\gamma_1, \gamma_2]$. $f(x)$ belongs to a sector whose boundaries are the straight lines $y = \gamma_1 x$ and $y = \gamma_2 x$. The following inequalities can be easily verified that $\forall x \in \mathbb{R}$:

- $\gamma_1\, |x| \le |f(x)| \le \gamma_2\, |x|$
- $\gamma_1\, x^2 \le x\, f(x) \le \gamma_2\, x^2$
- $\frac{\gamma_1}{\gamma_2^2} f^2(x) \le x f(x) \le \frac{\gamma_2}{\gamma_1^2} f^2(x)$.

Remark 1: All following results can be readily extended to more general models that $\dot{x}_i(t) = -\sum_{j=1}^{N} a_{ij} f_{ij}(x_i - x_j)$, i.e. different nonlinear functions for each edge, provided that $f_{ij} = f_{ji}$ holds, $\forall (i, j) \in E$.

3. Stability and convergence

Given system (1) and the control law (2), the resulting closed-loop system is given by

$$\dot{x}_i = -\sum_{j=1}^{N} a_{ij} f\big(x_i - x_j\big), \quad i \in \mathcal{V}. \tag{3}$$

First, we show that along the solution of system (3), the sum of agent states $\sum_{i=1}^{N} x_i$ remains constant. Its time derivative is given by $\sum_{i=1}^{N} \dot{x}_i = \sum_{i=1}^{N} \sum_{j=1}^{N} (a_{ij} - a_{ji})\, f(x_i - x_j) = 0$, due to the symmetric properties of both $f(\cdot)$ and $a_{ij} = a_{ji}$, $\forall i, j \in \mathcal{V}$. Denote the initial average of the states by $\frac{1}{N}\sum_{i=1}^{N} x_i(0) = C$. Then $\frac{1}{N}\sum_{i=1}^{N} x_i(t) = C$, $\forall t \ge 0$. We can compute the disagreement of each agent's state to this invariant centroid and use the norm of this disagreement vector as the Lyapunov function candidate:

$$V = \sum_{i=1}^{N} (x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)^2 = \sum_{i=1}^{N} (x_i - C)^2.$$

Its level sets $\Omega = \{\mathbf{x} \mid V \le c,\ c \in \mathbb{R}^+\}$ define compact sets in the agents' state space since $V \le c \Rightarrow C - \sqrt{c} \le x_i \le C + \sqrt{c}$, which is closed and bounded for $c > 0$, $\forall i \in \mathcal{V}$.

Theorem 3.1: *Assume the communication graph G remains static and undirected. Then the solution of the closed-loop system* (3) *converges to the invariant set* $\{\mathbf{x} \mid x_i = x_j,\ (i, j) \in E\}$. *Moreover, the average consensus can be reached exponentially if and only if G is connected.*

Proof: The time derivative of V along the trajectories of system (3) is given by

$$\begin{aligned}\dot{V} &= 2\left(\sum_{i=1}^{N} x_i \cdot \dot{x}_i - C\sum_{i=1}^{N} \dot{x}_i\right) = -2\sum_{i=1}^{N}\sum_{j=1}^{N} x_i \cdot a_{ij} f(x_i - x_j) \\ &= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[(x_i - x_j) f(x_i - x_j)] \\ &\le -\sum_{i=1}^{N}\sum_{j=1}^{N} \gamma_1\, a_{ij}(x_i - x_j)^2 \le 0.\end{aligned}$$

By LaSalle's Invariance principle Khalil (2002), asymptotic convergence to the invariant set $\{\mathbf{x} \mid \dot{V} = 0\}$ is guaranteed. $\dot{V} = 0$ implies that $a_{ij}(x_i - x_j)^2 = 0$, $i, j \in \mathcal{V}$. Since $a_{ij} = 0$, $\forall (i, j) \notin E$ and $a_{ij} > 0$, $(i, j) \in E$, the invariant set $\{\mathbf{x} \mid \dot{V} = 0\}$ is equivalent to $\{\mathbf{x} \mid x_i = x_j,\ (i, j) \in E\}$. If the underlying communication graph G is connected, it leads to the conclusion that all agents have the same states, which should coincide with the initial average, due to the fact that the average state is invariant. Furthermore, we can bound $\dot{V}$ in such a way that

$$\begin{aligned}\dot{V} &\le -\gamma_1 \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)^2 = -2\gamma_1\, \mathbf{x}^T L \mathbf{x} \\ &= -2\gamma_1\, (\mathbf{x} - C\,\mathbf{1})^T L\, (\mathbf{x} - C\,\mathbf{1}),\end{aligned}$$

where we used the facts that $L\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^T L = \mathbf{0}^T$. Assume that G is static and the Laplacian matrix is time-invariant. By the Courant-Fisher Theorem in Horn and Johnson (1990), since $\mathbf{1}^T(\mathbf{x} - C\mathbf{1}) = 0$, we have $(\mathbf{x} - C\mathbf{1})^T\ L\ (\mathbf{x} - C\mathbf{1}) \ge \lambda_2(L)\ (\mathbf{x} - C\mathbf{1})^T(\mathbf{x} - C\mathbf{1})$, where $\lambda_2(L)$ is the second largest eigenvalue of the Laplacian matrix, which is strictly positive for a connected G. Thus, $\dot{V} = -2\gamma_1\, (\mathbf{x} - C\mathbf{1})^T L\, (\mathbf{x} - C\mathbf{1}) \le -2\gamma_1\lambda_2(L)\ (\mathbf{x} - C\mathbf{1})^T(\mathbf{x} - C\mathbf{1}) = -2\gamma_1\lambda_2(L)\, V$. By applying the Comparison Lemma from Bacciotti and Rosier (2005), we get

$$\begin{aligned}V(t) &\le e^{-2\gamma_1\lambda_2(L)t}\, V(0) \Rightarrow |\mathbf{x}(t) - C\mathbf{1}| \\ &\le e^{-\gamma_1\lambda_2(L)t}\, |\mathbf{x}(0) - C\mathbf{1}|,\end{aligned}$$

which ensures all agents approach the average consensus exponentially fast, with the worst-case rate $\gamma_1\lambda_2(L)$. □

Since the agents in the network are moving, it is reasonable to assume that communication links can fail and be created. Namely, certain number of edges might be added or removed from the communication graph. We are interested in analysing whether it is possible to reach the average consensus in case of a network with switching topology.

Corollary 3.2: *Assume that $G(t)$ is time-varying but keeps connected. The closed-loop system* (3) *converges to the average consensus exponentially*.

Proof: Consider the same Lyapunov function candidate for system (3) under switching topologies. Since V does not depend on the network topology G, V is continuous over the switching instants. We have shown that V is non-increasing with static topologies. Thus, we can conclude that V is non-increasing along the solutions of the switching system. Denote the sequence of underlying switching topologies as $\{G_k\}$, $k = 1, 2, \ldots$. Since G_k is always connected, following Theorem 2.1 in Liberzon (2003) and Theorem 3.1 in this paper, it holds that

$$\dot{V} \leq -2\gamma_1 \lambda_2(L)\, V \leq -2\gamma_1\, \lambda_2^*\, V,$$

where $\lambda_2^* = \min_k\{\lambda_2(L(G_k))\}$, i.e. the minimal second largest eigenvalue of the Laplacian matrices corresponding to all graphs in $\{G_k\}$. Thus, V is a valid common Lyapunov function for the switching system. Moreover the disagreement vector globally converges to the origin by $|\mathbf{x}(t) - C\mathbf{1}| \leq e^{-\gamma_1 \lambda_2^* t}|\mathbf{x}(0) - C\mathbf{1}|$, where $\gamma_1 \lambda_2^*$ is the worst-case rate for all connected graphs with N vertices. □

4. Sampled system

The stability and performance of continuous system (3) was discussed in the previous section. In many cases, in order to reduce the control effort and favour digital implementations, the control laws are updated periodically according to a constant sampling period T_s. Moreover, it is well known that the value of the sampling period greatly affects the stability and performance of the resulting discrete-time system. While system (3) is a continuous process, we can sampled control law (2) with sampling period $T_s > 0$, so that

$$u_i(t) = -\sum_{j=1}^{N} a_{ij} f(x_i(kT_s) - x_j(kT_s)), \quad t \in [kT_s, kT_s + T_s). \tag{4}$$

Then the discrete-time closed-loop system is given by

$$x_i(k+1) = x_i(k) - \sum_{j=1}^{N} a_{ij} f(x_i(k) - x_j(k))\, T_s\,,$$

where we simplify the notation kT_s by k because all agents share the same sampling clock. Similarly it can be verified that the sum of states $\sum_{i=1}^{N} x_i(k)$ remains invariant and the invariant centroid is denoted by C as in the continuous case.

Consider the discrete-time Lyapunov function candidate $V(k) = \sum_{i=1}^{N}(x_i(k) - C)^2$. The level sets of V are closed and bounded as before. Since $V(k) = \sum_{i=1}^{N} x_i^2(k) - NC^2$, the difference $V(k+1) - V(k)$ is computed as

$$\begin{aligned}
&V(k+1) - V(k)\\
&= \sum_{i=1}^{N} [x_i(k+1) + x_i(k)][x_i(k+1) - x_i(k)]\\
&= \sum_{i=1}^{N} \left[2x_i(k) - \sum_{j=1}^{N} a_{ij} f(x_i(k) - x_j(k))\, T_s \right]\\
&\quad \times \left[-\sum_{j=1}^{N} a_{ij} f(x_i(k) - x_j(k))\, T_s \right]\\
&= -T_s \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\,[x_i(k) - x_j(k)] f(x_i(k) - x_j(k))\\
&\quad + T_s^2 \sum_{i=1}^{N} \left[\sum_{j=1}^{N} a_{ij} f(x_i(k) - x_j(k)) \right]^2,
\end{aligned} \tag{5}$$

where in the last equation we use the symmetric property of both $f(x)$ and a_{ij}.

Theorem 4.1: *Assume G is static and undirected. The solution of the multi-agent system* (1) *under sampled control law* (4) *is guaranteed to converge to the average consensus asymptotically if G is connected and the sampling time T_s satisfies*

$$0 < T_s < \frac{\gamma_1}{\gamma_2^2} \frac{1}{\max_i\{\sum_{j=1}^{N} a_{ij}\}}.$$

Proof: Applying the Cauchy–Schwarz inequality to the second part of (5), we get

$$\begin{aligned}
&\left[\sum_{j=1}^{N} a_{ij} f(x_i(k) - x_j(k)) \right]^2\\
&= \left[\sum_{j=1}^{N} \sqrt{(a_{ij})}\, \sqrt{(a_{ij})} f(x_i(k) - x_j(k)) \right]^2\\
&\leq \sum_{j=1}^{N} a_{ij} \sum_{j=1}^{N} a_{ij}[f(x_i(k) - x_j(k))]^2.
\end{aligned}$$

Moreover since $x \cdot f(x) \geq \frac{\gamma_1}{\gamma_2^2}[f(x)]^2$, $\forall x \in \mathbb{R}$, which can be applied to the first part of (5), we get

$$\begin{aligned}
V(k+1) - V(k) \leq &-\frac{\gamma_1}{\gamma_2^2} T_s \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}[f(x_i(k) - x_j(k))]^2\\
&+ T_s^2 \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} \sum_{j=1}^{N} a_{ij}[f(x_i(k) - x_j(k))]^2.
\end{aligned}$$

Denote by $z_i = \sum_{j=1}^{N} a_{ij}\,[f(x_i(k) - x_j(k))]^2 \geq 0$ and d_i is the *degree* of agent i defined earlier. We have

$$\begin{aligned} V(k+1) - V(k) &\leq -\frac{\gamma_1}{\gamma_2^2} T_s \sum_{i=1}^{N} z_i + T_s^2 \sum_{i=1}^{N} d_i z_i \\ &\leq -\frac{\gamma_1}{\gamma_2^2} T_s \sum_{i=1}^{N} z_i + T_s^2 \max_i\{d_i\} \\ &\quad \times \sum_{i=1}^{N} z_i = T_s\left(T_s \max_i\{d_i\} - \frac{\gamma_1}{\gamma_2^2}\right)\sum_{i=1}^{N} z_i. \end{aligned}$$

If it holds that

$$0 < T_s < \frac{\gamma_1}{\gamma_2^2}\frac{1}{\max_i\{d_i\}} = \frac{\gamma_1}{\gamma_2^2}\frac{1}{\max_i\{\sum_{j=1}^{N} a_{ij}\}},$$

where $\max_i\{d_i\}$ is the maximum degree of all agents, then $V(k+1) - V(k) \leq 0$, where the equality $V(k+1) = V(k)$ holds only when $z_i = 0$, $\forall i \in \mathcal{V}$. Since $z_i = \sum_{j=1}^{N} a_{ij}\,[f(x_i(k) - x_j(k))]^2 \geq 0$ and $f(x) = 0$ only if $x = 0$, it implies that $x_i = x_j$, $\forall (i, j) \in E$. Moreover, as the underlying graph G is connected, it leads to the fact that $x_1 = \cdots = x_N$, which should coincide with the invariant average state. Application of LaSalle's Invariance principle for discrete time systems from Lygeros, Johansson, Simic, Zhang, and Sastry (2003), establishes the convergence of system (4) to the average consensus as $t \to \infty$. □

Moreover by Geršgorin Disk theorem in Olfati-Saber and Murray (2004), the maximal eigenvalues of the Laplacian matrix $\lambda_{\max}(L)$ should satisfy $|\lambda_{\max}(L) - \max_i\{d_i\}| \leq \max_i\{d_i\}$, which implies $\max_i\{d_i\} \geq \frac{\lambda_{\max}(L)}{2}$. Thus it holds that

$$0 < T_s \leq \frac{\gamma_1}{\gamma_2^2}\frac{1}{\max_i\{d_i\}} \leq \frac{\gamma_1}{\gamma_2^2}\frac{2}{\lambda_{\max}(L)}.$$

As expected, the constraint on the sampling time is tighter than the linear counterparts discussed in Xie et al. (2009) when $\gamma_2 > 1$ and $\gamma_1 < \gamma_2$.

Remark 2: The above conclusions can also be extended to switching topologies by applying similar arguments as in Corollary 3.2. The Lyapunov function candidate V can serve as a common Lyapunov function for the sampled system under switching topologies.

5. Aperiodic control design

In this section, we relax the requirement for periodic sampling at pre-specified instants by considering aperiodic event-triggered control strategy. An event-driven approach seems more suitable in order to allow more agents into the system without increasing the communication and computational cost, as discussed in Astrom and Bernhardsson (2002). In particular, the distributed event-triggered rules given in Dimarogonas et al. (2012) enforce each agent to update its control input whenever a certain state error measurement threshold is violated, as well as when the control law of its neighbours is updated. Moreover, the communication among the neighbouring agents can also be event-triggered as proposed in Seyboth et al. (2011). Whenever one agent sends or receives a new state measurement, its control law is updated, which renders that both communication and control-law update are event-triggered and they obey the same triggering conditions.

In the sequel, we use the same notations as in Dimarogonas et al. (2012). Assume there exists a separate sequence of events defined for each agent i, occurring at times $t_1^i, t_2^i, \ldots, t_k^i$, $k = 1, 2, \ldots$. An event for agent i is triggered as soon as the triggering condition (12) is fulfilled. The control strategy for agent i is given by

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} a_{ij} f\Big(x_i(t_k^i) - x_j(t_{k'(t)}^j)\Big), \quad i \in \mathcal{V}, \tag{6}$$

where $k'(t) = \arg\min_{l \in \mathbb{N}:\, t \geq t_l^j}\{t - t_l^j\}$ is the last event time of agent j. At agent i's event time t_k^i, it updates its control law (6) by letting $x_i(t_k^i) = x_i(t)$ while at the same time it broadcasts its latest state $x_i(t_k^i)$ to all its neighbours $j \in \mathcal{N}_i$. Consequently, any agent $j \in \mathcal{N}_i$ updates its control law by letting $x_i(t_{k'(t)}^i) = x_i(t_k^i)$. Thus, the decentralised control law (6) for each agent is updated both at its own event times as well as the event times of its neighbours. Namely, the time instants when agent i updates its control law (6) are $\{t_k^i\} \cup \{t_k^j\}$, $\forall j \in \mathcal{N}_i$.

Before we present the triggering condition, some useful variables are introduced. The state measurement error for agent i is defined as

$$e_i(t) = x_i(t_k^i) - x_i(t), \tag{7}$$

where $t \in [t_k^i,\ t_{k+1}^i)$. $e_i(t)$ is set to zero whenever $t \in \{t_k^i\}$, $k \in \mathbb{Z}$. The control law (6) indicates that each agent takes into account the latest state information of its neighbours in the control law. The closed-loop system is given by

$$\begin{aligned} \dot{x}_i &= -\sum_{j \in \mathcal{N}_i} a_{ij} f\big(x_i(t_k^i) - x_j(t_{k'}^j)\big) \\ &= -\sum_{j \in \mathcal{N}_i} a_{ij} f\Big(x_i(t) - x_j(t) + (e_i(t) - e_j(t))\Big). \end{aligned} \tag{8}$$

Since $\frac{1}{N}\sum_{i=1}^{N} \dot{x}_i(t) = -\sum_{i=1}^{N}\sum_{j \in \mathcal{N}_i} a_{ij} f\big(x_i(t) - x_j(t) + (e_i(t) - e_j(t))\big) = 0$, the average of the states is still invariant, i.e. $\frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) = C$.

We consider the same Lyapunov function candidate $V=\sum_{i=1}^{N}(x_i-C)^2$. The level sets of V are compact as before. In the sequel, we omit the time index t for brevity. The time derivative of V along the trajectory of system (8) is given by

$$\begin{aligned}\dot V &= -\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i-x_j)f(x_i-x_j+e_i-e_j)\\ &= -\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i-x_j+e_i-e_j)f(x_i-x_j+e_i-e_j)\\ &\quad+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}(e_i-e_j)f(x_i-x_j+e_i-e_j)\\ &\le -\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}\frac{\gamma_1}{\gamma_2^2}f^2(x_i-x_j+e_i-e_j)\\ &\quad+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}(|e_i|+|e_j|)\,|a_{ij}f(x_i-x_j+e_i-e_j)|.\end{aligned}$$

Denoting $z_{ij}=a_{ij}\ f(x_i-x_j+e_i-e_j)$ and using the inequality $|xy|\le\frac{a}{2}x^2+\frac{1}{2a}y^2$, $a>0$, we can bound $\dot V$ as

$$\dot V\le\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left(a-\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i\right)|z_{ij}|^2+\sum_{i=1}^{N}\frac{1}{a}|\mathcal{N}_i||e_i|^2, \qquad (9)$$

where $\overline{\tau}_i=\frac{1}{\max_{j\in\mathcal{N}_i}\{a_{ij}\}}$. By enforcing the condition

$$e_i^2\le\frac{\rho_i\,a(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a)}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}z_{ij}^2+\varepsilon_i,\quad i\in\mathcal{V}, \qquad (10)$$

where $0<a<\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i$, $0<\rho_i<1$ and $\varepsilon_i>0$ are scalars, with (9) we get

$$\dot V\le-\sum_{i=1}^{N}\left((1-\rho_i)\left(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a\right)\sum_{j\in\mathcal{N}_i}z_{ij}^2-\frac{1}{a}|\mathcal{N}_i|\varepsilon_i\right),$$

which is negative definite if

$$\sum_{i=1}^{N}(1-\rho_i)\left(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a\right)\sum_{j\in\mathcal{N}_i}z_{ij}^2>\sum_{i=1}^{N}\frac{1}{a}|\mathcal{N}_i|\varepsilon_i. \qquad (11)$$

Let each agent i update the control input (6) whenever the triggering condition

$$e_i^2\ge\frac{\rho_i\,a\left(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a\right)}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}z_{ij}^2+\varepsilon_i \qquad (12)$$

is fulfilled, in addition to its neighbours' event times. It should be emphasised that the composing elements of the triggering condition (12), $z_{ij}=a_{ij}f(x_i-x_j+e_i-e_j)=a_{ij}f(x_i(t_k^i)-x_j(t_{k'}^j))$ are constant between two consecutive control law updating events, which means the right-hand side of (12) is piecewise-constant. In addition, it is worth mentioning that (12) can be evaluated only by locally available measurements.

Theorem 5.1: *Consider the first-order system* (1) *with the control law* (6) *and updating rule* (12). *Assume the underlying graph G is connected,* $0<a<\frac{\gamma_1}{\gamma_2^2}\frac{1}{\max_{(i,j)}\{a_{ij}\}}$, $\varepsilon_i>0$ *and* $0<\rho_i<1$, $\forall i\in\mathcal{V}$. *Then for any initial condition the solution of the closed-loop system* (8) *converges to the invariant set* (15). *Moreover, the inter-event interval is strictly lower bounded by the positive number* (16).

Proof: The Lyapunov function candidate V is positive definite, smooth and regular. $V=0$ when all states equal to the initial average. The level sets of V define compact set with respect to agents' states. In particular, $V\le c$ implies $|x_i-C|\le c\Rightarrow C-\sqrt{c}\le x_i\le C+\sqrt{c}$, $\forall i\in\mathcal{V}$ and $c>0$. Moreover, we have shown that $\dot V\le 0$ as long as the triggering condition (12) is enforced and (11) holds. Regarding (11), when

$$\sum_{i=1}^{N}(1-\rho_i)\left(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a\right)\sum_{j\in\mathcal{N}_i}z_{ij}^2\le\sum_{i=1}^{N}\frac{1}{a}|\mathcal{N}_i|\varepsilon_i, \qquad (13)$$

we can bound V that

$$\begin{aligned}V(t)&=\sum_{i=1}^{N}(x_i-C)^2\le\frac{2}{\lambda_2(L)}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a_{ij}[(x_i-x_j+e_i-e_j)^2\\ &\quad+(e_i-e_j)^2]\\ &=\frac{2}{\lambda_2(L)}\left[\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a_{ij}(x_i-x_j+e_i-e_j)^2\right.\\ &\quad\left.+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a_{ij}(e_i-e_j)^2\right]\\ &\le\frac{2}{\lambda_2(L)}\left[\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\underline{\tau}}{\gamma_1^2}z_{ij}^2+\lambda_{\max}(L)\sum_{i=1}^{N}e_i^2\right],\end{aligned} \qquad (14)$$

where $\underline{\tau}=\frac{1}{\min_{(i,j)}\{a_{ij}\}}$ and we use the fact that $\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a_{ij}(e_i-e_j)^2=\mathbf{e}^TL\mathbf{e}\le\lambda_{\max}(L)\,\mathbf{e}^T\mathbf{e}$.

By (13), the first term of (14) is bounded by

$$\frac{\underline{\tau}}{\gamma_1^2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}z_{ij}^2\le\frac{\underline{\tau}}{\min_i\{(1-\rho_i)(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a)\}\gamma_1^2}\sum_{i=1}^{N}\frac{1}{a}|\mathcal{N}_i|\varepsilon_i=M_1.$$

The second term is bounded by the condition (10), namely

$$\begin{aligned}\sum_{i=1}^{N}e_i^2&\le\sum_{i=1}^{N}\left(\frac{\rho_i\,a(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a)}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}z_{ij}^2+\varepsilon_i\right)\\ &\le\max_i\left\{\frac{\rho_i\,a(\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i-a)}{|\mathcal{N}_i|}\right\}\frac{\gamma_1^2}{\underline{\tau}}M_1+\sum_{i=1}^{N}\varepsilon_i=M_2.\end{aligned}$$

Consequently, (13) implies that the level set of V is bounded by

$$V(t) \le \frac{2}{\lambda_2(L)}(M_1 + \lambda_{\max}(L)M_2). \tag{15}$$

To conclude, since $V > \frac{2}{\lambda_2(L)}(M_1 + \lambda_{\max}(L)M_2)$ implies (11) and $\dot{V} \le 0$, the solution of system (8) reaches the invariant set (15) and remains inside. Note that M_1, $M_2 \to 0$ when $\varepsilon_i \to 0$.

Next, we show that the inter-event time is lower bounded by a strictly positive lower bound. Applying similar arguments as in the proof of Theorem 1, Dimarogonas (2011), the minimum lower bound is given by $t^i_{k+1} - t^i_k \ge \frac{\sqrt{\varepsilon_i}}{\bar{u}_i}$, where $\bar{u}_i$ is an upper bound on the admissible control input (6) of agent i. In our case, $\bar{u}_i$ is determined by

$$\begin{aligned}
|u_i(t)| &= \left| -\sum_{j\in\mathcal{N}_i} a_{ij} f\Big(x_i(t^i_k) - x_j(t^j_{k'(t)})\Big)\right| \\
&\le \frac{1}{\underline{\tau}_i}\sum_{j\in\mathcal{N}_i}\left| f\Big(x_i(t^i_k) - x_j(t^j_{k'(t)})\Big)\right| \\
&\le \frac{\gamma_2}{\underline{\tau}_i}\sum_{j\in\mathcal{N}_i}\left|\Big(x_i(t^i_k) - x_j(t^j_{k'(t)})\Big)\right| \\
&\le \frac{\gamma_2}{\underline{\tau}_i}\sqrt{|\mathcal{N}_i|\sum_{j\in\mathcal{N}_i}\Big(x_i(t^i_k) - x_j(t^j_{k'(t)})\Big)^2} \\
&\le \frac{\gamma_2}{\underline{\tau}_i}\sqrt{|\mathcal{N}_i|\sum_{i=1}^{N}\sum_{j=1}^{N}\Big(x_i(t^i_k) - x_j(t^j_{k'(t)})\Big)^2} \\
&= \frac{\gamma_2}{\underline{\tau}_i}\sqrt{|\mathcal{N}_i|\, V(t^i_k)\cdot 2N} \le \frac{\gamma_2}{\underline{\tau}_i}\sqrt{2N|\mathcal{N}_i|\, V(0)} = \bar{u}_i,
\end{aligned}$$

where we use the fact $V = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_j)^2$ and $|f(x)| \le \gamma_2|x|$, $\forall x \in \mathbb{R}$. Thus for agent i, the inter-event time should satisfy

$$t^i_{k+1} - t^i_k \ge \frac{\sqrt{\varepsilon_i}}{\bar{u}_i} \ge \frac{\underline{\tau}_i\sqrt{\varepsilon_i}}{\gamma_2\sqrt{2N|\mathcal{N}_i|\, V(0)}}, \quad i \in \mathcal{V}. \tag{16}$$

Thus, the inter-event interval of all agents are lower bounded by a strictly positive number. □

Remark 3: The extension of Theorem 5.1 to switching topologies is possible, given the assumption that each agent has the knowledge of when the underlying communication topology switches. Then those switching instants can be added to every agent's triggering sequence, which means that all agents update their control law simultaneously at the switching instants.

6. $\mathcal{L}_2$ gain analysis

Inspired by the work in Dimarogonas (2011), we would like to examine the robustness of the proposed event-triggered strategy (12) with respect to additive disturbances in the system model. In particular, we assume that each agent's dynamics are perturbed by an additive noise that

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_i} a_{ij} f\big(x_i(t^i_k) - x_j(t^j_{k'})\big) + w_i, \quad i \in \mathcal{V}, \tag{17}$$

where each $w_i \in \mathbb{R}$ is a one-dimensional $\mathcal{L}_2$ function and is uniformly bounded as $|w_i| < \bar{w} < \infty$, $\forall i$. We consider the same Lyapunov function candidate $V = \sum_{i=1}^{N}(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)^2$, but this time $\frac{1}{N}\sum_{i=1}^{N} x_i$ is not constant anymore since $\sum_{i=1}^{N}\dot{x}_i = \sum_{i=1}^{N} w_i$ is not necessarily zero. The time derivative of V along the trajectories of system (17) is given by

$$\begin{aligned}
\dot{V} = &-\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) f(x_i - x_j + e_i - e_j) \\
&+ 2\sum_{i=1}^{N}\left(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i\right) w_i.
\end{aligned}$$

It has been shown in (9) that the first term is bounded by $\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}(a - \frac{\gamma_1}{\gamma_2^2}\bar{\tau}_i)|z_{ij}|^2 + \sum_{i=1}^{N}\frac{1}{a}|\mathcal{N}_i||e_i|^2$. The second term fulfills $2\sum_{i=1}^{N}(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)\, w_i \le \zeta\sum_{i=1}^{N}(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)^2 + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2$, where $\zeta > 0$. Moreover, following the same calculation in (14) we can show that $\sum_{i=1}^{N}(x_i - \frac{1}{N}\sum_{i=1}^{N} x_i)^2 \le \frac{2}{\lambda_2(L)}[\sum_{i=1}^{N}\frac{\underline{\tau}_i}{\gamma_1^2}\sum_{j\in\mathcal{N}_i} z_{ij}^2 + \lambda_{\max}(L)\sum_{i=1}^{N} e_i^2]$ where $\underline{\tau}_i = \frac{1}{\min_{j\in\mathcal{N}_i}\{a_{ij}\}}$. Thus, $\dot{V}$ can be bounded as

$$\begin{aligned}
\dot{V} \le &\sum_{i=1}^{N}\left(a - \frac{\gamma_1}{\gamma_2^2}\bar{\tau}_i + \frac{2\zeta\,\underline{\tau}_i}{\lambda_2(L)\gamma_1^2}\right)\sum_{j\in\mathcal{N}_i} z_{ij}^2 \\
&+ \sum_{i=1}^{N}\left(\frac{1}{a}|\mathcal{N}_i| + \frac{2\lambda_{\max}(L)}{\lambda_2(L)}\zeta\right) e_i^2 + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2.
\end{aligned}$$

Denote by $\alpha_i = \frac{\gamma_1}{\gamma_2^2}\bar{\tau}_i - a - \frac{2\zeta\,\underline{\tau}_i}{\lambda_2(L)\gamma_1^2}$ and $\beta_i = \frac{1}{a}|\mathcal{N}_i| + \frac{2\lambda_{\max}(L)}{\lambda_2(L)}\zeta$, $\forall i \in \mathcal{V}$. Assume that $0 < a < \left(\frac{\gamma_1}{\gamma_2^2}\bar{\tau}_i - \frac{2\zeta\underline{\tau}_i}{\lambda_2(L)\gamma_1^2}\right)$ so that $\alpha_i > 0$, $\forall i \in \mathcal{V}$. By enforcing the condition

$$e_i^2 \le \frac{\rho_i\,\alpha_i}{\beta_i}\sum_{j\in\mathcal{N}_i} z_{ij}^2 + \varepsilon_i, \tag{18}$$

where $0 < \rho_i < 1$ and $\varepsilon_i > 0$ are scalars, we have

$$\begin{aligned}
\dot{V} &\le -\sum_{i=1}^{N}(1-\rho_i)\alpha_i\sum_{j\in\mathcal{N}_i} z_{ij}^2 + \sum_{i=1}^{N}\beta_i\,\varepsilon_i + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2 \\
&\le -\sum_{i=1}^{N}\frac{(1-\rho_i)\alpha_i}{|\mathcal{N}_i|}\left(\sum_{j\in\mathcal{N}_i} z_{ij}\right)^2 + \sum_{i=1}^{N}\beta_i\,\varepsilon_i + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2 \\
&\le -\min_i\left\{\frac{(1-\rho_i)\alpha_i}{|\mathcal{N}_i|}\right\}\|z\|^2 + \frac{1}{\zeta}\|w\|^2 + \sum_{i=1}^{N}\beta_i\,\varepsilon_i,
\end{aligned}$$

where **z**, **w** represent the stack vector composed of u_i and w_i respectively. Based on Wang and Lemmon (2009), the closed-loop system (17) is finite $\mathcal{L}_2$-gain stable with an induced gain less than $\frac{1}{\sqrt{\zeta \min_i\{\frac{(1-\rho_i)\alpha_i}{|\mathcal{N}_i|}\}}}$. Moreover it is guaranteed that $\dot{V} \leq 0$ as long as

$$\sum_{i=1}^{N}(1-\rho_i)\alpha_i \sum_{j\in\mathcal{N}_i} z_{ij}^2 > \sum_{i=1}^{N}\beta_i\,\varepsilon_i + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2. \tag{19}$$

Consequently, by (18) each agent i updates the control input (17) whenever the condition

$$e_i^2 \geq \frac{\rho_i\,\alpha_i}{\beta_i}\sum_{j\in\mathcal{N}_i} z_{ij}^2 + \varepsilon_i \tag{20}$$

is fulfilled.

Theorem 6.1: *Consider the closed-loop system* (17) *with the updating rule* (20). *Assume the underlying graph* G *is connected*, $0 < a < (\frac{\gamma_1}{\gamma_2^2}\overline{\tau}_i - \frac{2\zeta\underline{\tau}_i}{\lambda_2(L)\gamma_1^2})$, $\varepsilon_i > 0$ *and* $0 < \rho_i < 1$, $\forall i \in \mathcal{V}$. *Then, for any initial condition its solution converges to the invariant set* (21). *Moreover, the inter-event triggering interval is strictly lower bounded by the positive number* (22).

Proof: Similar analyses as in the proof of Theorem 5.1 are applies here. When (19) is not fulfilled, $\sum_{i=1}^{N}(1-\rho_i)\alpha_i\sum_{j\in\mathcal{N}_i} z_{ij}^2 \leq \sum_{i=1}^{N}\beta_i\,\varepsilon_i + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2$, which implies $\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} z_{ij}^2 \leq \frac{1}{\min_i\{(1-\rho_i)\alpha_i\}}\sum_{i=1}^{N}\beta_i\,\varepsilon_i + \frac{1}{\zeta}\sum_{i=1}^{N} w_i^2 = M_3$. By (14) and (18), we can bound V as follows

$$\begin{aligned} V(t) &\leq \frac{2}{\lambda_2(L)}\left[\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\underline{\tau}}{\gamma_1^2} z_{ij}^2 + \lambda_{\max}(L)\sum_{i=1}^{N} e_i^2\right] \\ &\leq \frac{2}{\lambda_2(L)}\left[\frac{\underline{\tau}}{\gamma_1^2}M_3 + \lambda_{\max}(L)\left(\max_i\{\frac{\rho_i\alpha_i}{\beta_i}\}M_3 + \sum_i \varepsilon_i\right)\right] \\ &= M_4. \end{aligned} \tag{21}$$

As a result, since $V(t) > M_4$ implies (19) and $\dot{V} \leq 0$, the solution of system (17) reaches the invariant set (21) and remains inside. Note that the above set converges to zero when ε_i, $w_i \to 0$.

As stated in Theorem 5.1, the minimum lower bound on the inter-event time is given by $t_{k+1}^i - t_k^i \geq \frac{\sqrt{\varepsilon_i}}{\bar{u}_i}$, where $\bar{u}_i$ is an upper bound on the admissible control input (17) of agent i. In the presence of additive noise, $\bar{u}_i$ is given by

$$\begin{aligned} |u_i(t)| &= \left| -\sum_{j\in\mathcal{N}_i} a_{ij} f\Big(x_i(t_k^i) - x_j(t_{k'(t)}^j)\Big) + w_i \right| \\ &\leq \frac{1}{\underline{\tau}_i}\sum_{j\in\mathcal{N}_i}\Big| f\Big(x_i(t_k^i) - x_j(t_{k'(t)}^j)\Big)\Big| + |w_i| \\ &\leq \frac{\gamma_2}{\underline{\tau}_i}\sqrt{2N|\mathcal{N}_i|\,V(0)} + \bar{w} = \bar{u}_i, \end{aligned}$$

where we use the same techniques as deriving (16) and $\bar{w} = \max_i\{w_i\} < \infty$. Thus, for each agent i, the inter-event interval should satisfy

$$t_{k+1}^i - t_k^i \geq \frac{\sqrt{\varepsilon_i}}{\bar{u}_i} \geq \frac{\sqrt{\varepsilon_i}}{\frac{\gamma_2}{\underline{\tau}_i}\sqrt{2N|\mathcal{N}_i|\,V(0)} + \bar{w}}, \quad i \in \mathcal{V}. \tag{22}$$

The lower bound is a strictly positive number as $V(0)$, $\bar{w} < \infty$. This completes the proof. □

7. Example

In this section, we provide numerical simulations to support our results. We consider a network of four first-order agents moving in x-coordinates. The neighbouring sets are given by $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \{1,3\}$, $\mathcal{N}_3 = \{2,4\}$ and $\mathcal{N}_4 = \{1,3\}$. Furthermore, the nonlinear function f: $f(x) = x\ (2 - \cos^2 x)$, $x \in \mathbb{R}$ is chosen.

As illustrated in Figure 1, it satisfies that $\gamma_1 = 1$ and $\gamma_2 = 2$. Four first-order agents start from arbitrary initial conditions and evolve under the event-triggered control system (6) and (12), aiming at the average consensus in x-coordinates. We assume that $\rho_i = 0.9$ and $\varepsilon_i = 0.1$ for each agent i. Figure 2 shows the trajectories of four agents, where the red dots denote the event times when the triggering condition (12) at each agent is satisfied. It can be seen that the inter-event interval is strictly positive. Figure 3 illustrates the evolution of the measurement error (7) in red and the triggering condition (12) in blue. It can be seen that the boundary of the triggering condition is piecewise constant and the measurement error is reset to zero when it meets the boundary. The observation that the boundary converges to a positive constant as $t \to \infty$ is due to fact that the first term of (12) approaches zero when $|x_i - x_j| \to 0$ and ε_i is a constant design parameter. On the other hand, Figures 4 and 5 show the same multi-agent system but perturbed by additive noises $\mathbf{w} = [0.01\ \ 0.03\ \ 0.02\ \ 0.01]$ as in (17). Similar performances are achieved by applying the event-triggered law (17) and (20). An invariant set close to the average consensus is obtained and the inter-event interval is shown to be strictly lower bounded.

It is worth mentioning that ρ_i and ε_i are the key control-design parameters in our event-triggered control laws. Clearly, the upper bound for $|e_i|$ from (12) and (20) becomes tighter when we decrease ρ_i or ε_i, which leads to more frequent triggering and shorter inter-event intervals, and consequently more control effort. On the other hand, the invariant set around the average consensus becomes smaller when ρ_i or ε_i are decreased in (15) and (21). Thus, the choice of ρ_i and ε_i

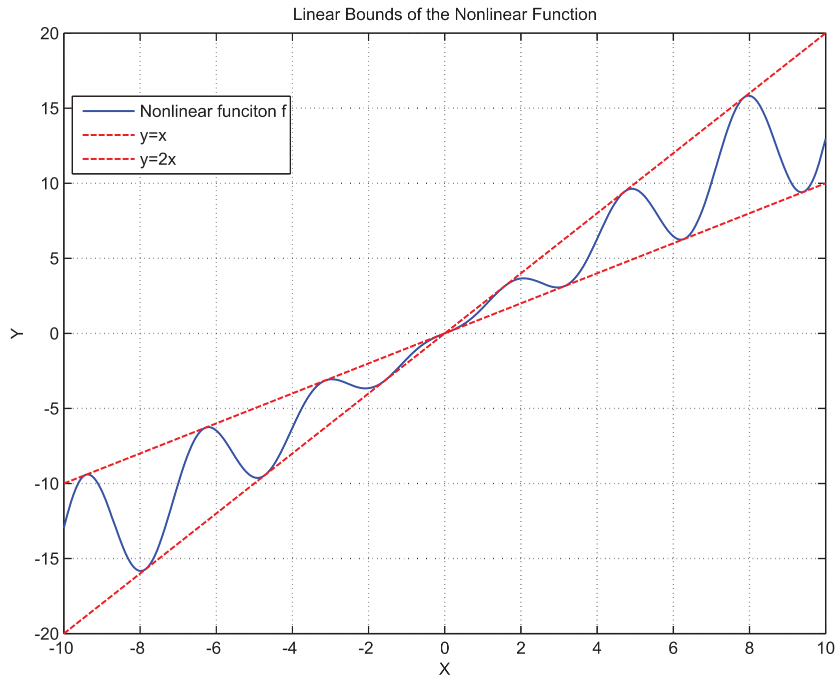

Figure 1. $f(\cdot)$ belongs to the sector $[\gamma_1, \gamma_2]$.

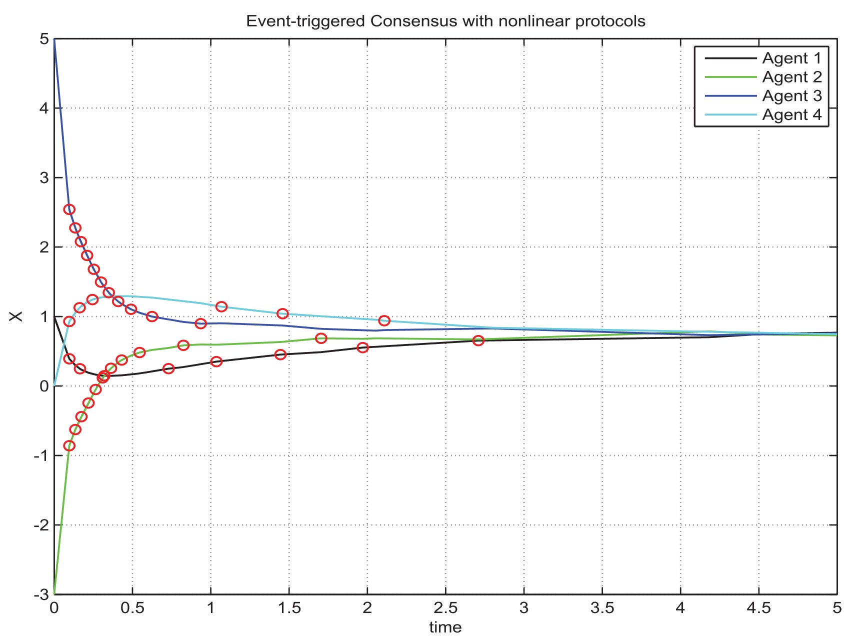

Figure 2. Trajectories of four agents, red dots denote the triggered instants.

is to determine the trade-off between consensus precision and control effort.

8. Conclusions

In this article, we take into account the first-order multi-agent system with a generic class of nonlinear control laws. We first analyse the stability of the continuous system and then extend the results to the same system with periodically sampled inputs. Explicit upper bounds on sampling intervals are derived to ensure global convergence. Moreover, we design a decentralised event-triggered law that substantially reduces both the control efforts

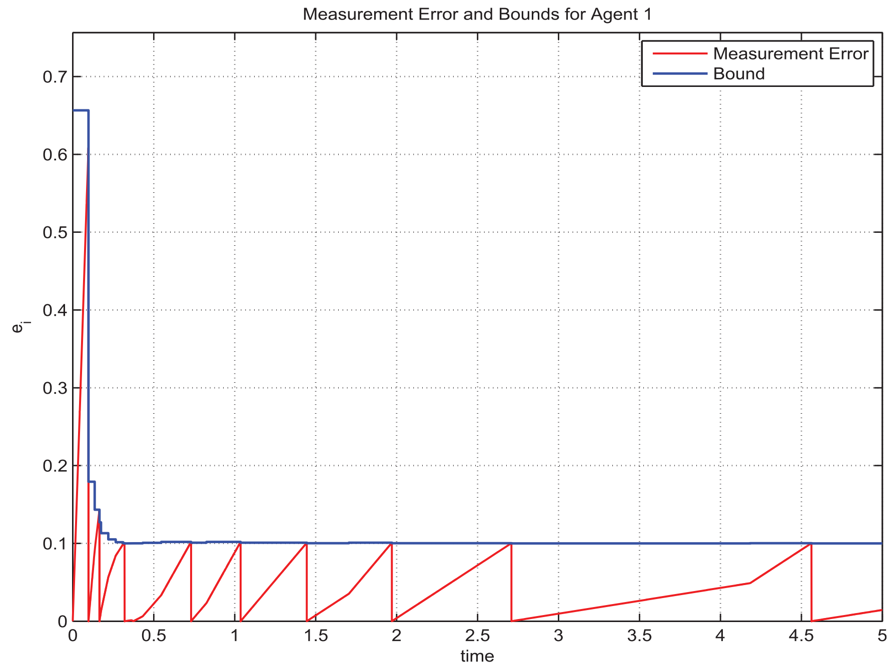

Figure 3. Measurement error and the event-trigger condition (the upper line) of agent 1.

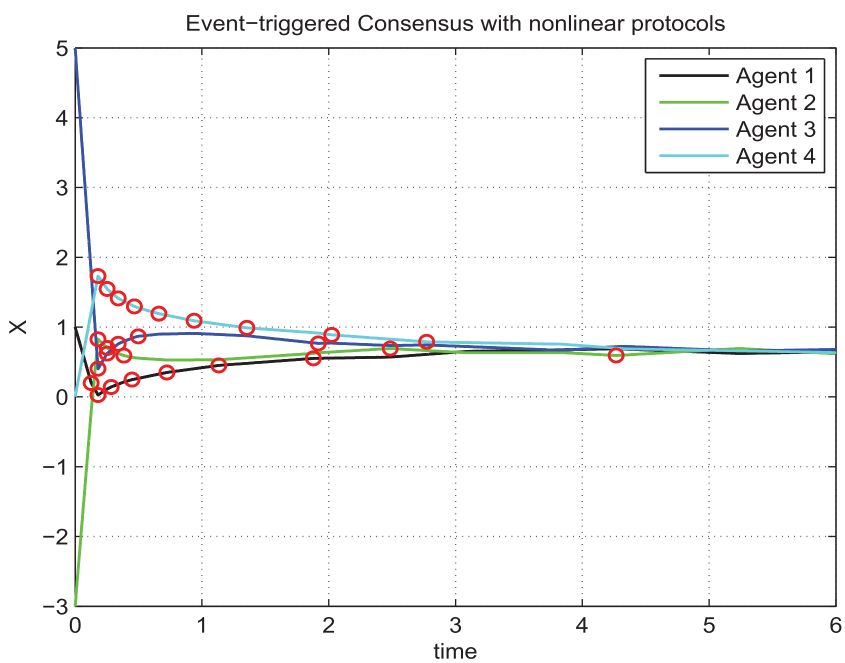

Figure 4. Trajectories of four agents with additive noises.

and the communication efforts. At the same time, the case of infinitely frequent triggering is excluded by showing that the inter-event interval is strictly larger than a positive lower bound. Moreover, the robustness of the event-triggered control law with respect to additive disturbances is examined by applying $\mathcal{L}_2$ gain analysis in the last part. Further research could take into account the same multi-agent system under directed communication topologies and other cooperative tasks than consensus.

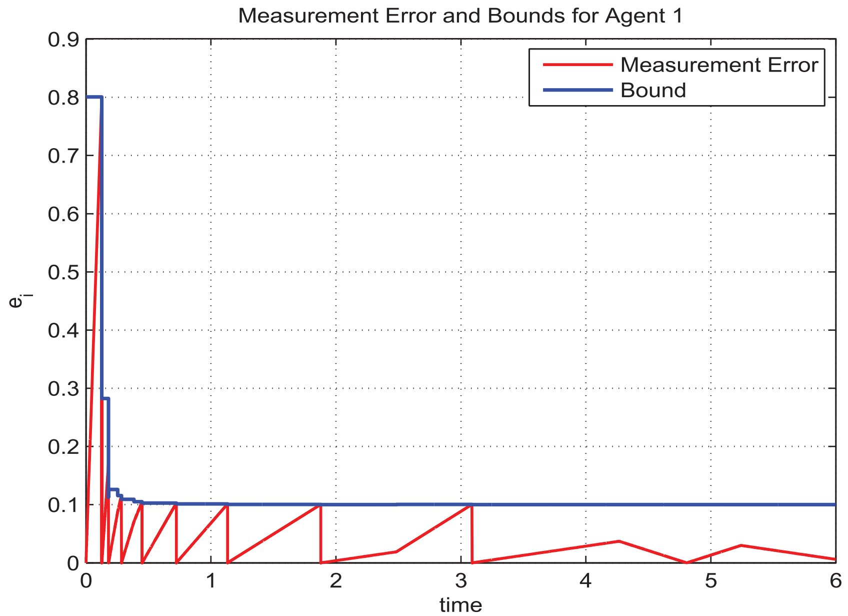

Figure 5. Measurement error and the event-trigger condition (the upper line) of agent 1.

Acknowledgements

This work was supported by the Swedish Research Council(VR) and the EU HYCON2 NoE. The second author is also affiliated with the Center of Autonomous Systems and is supported by the VR contract 2009-3948.

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